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ROOTS OF MATHEMATICS*

Abstract. To understand the origin of mathematics it seems reasonable to turn our attention to its deep anthropological roots and thus to consider the role of the central nervous system; the evolution of the faculties of thinking, speaking, and understanding symbols; the emergence of mythology, magic and rites; and early geometrical and arithmetical conceptions and their initial development. The article offers a survey of our knowledge on these roots, based upon archaeology, ethnology, psychology, linguistics, cognitive sciences, and it underlines the enormous amount of work done in elaborating primary mathematical ideas in prehistoric times.

Keywords: central nervous system, thinking, speaking, symbols, mythology, primary geometry, primary arithmetic.

1. Introduction

The primary motivation for dealing with the present topic was the initial chapter of Penrose's book *The Road to Reality*[1]. Surprised to see how little we know about the *roots of science* (chapter 1 of that book), I have undertaken the task of looking for a while upon the *roots of mathematics*.

The history of mathematics shows that some five thousands years ago, during the emerging first historic civilizations, mathematics was already present. Among the oldest preserved written texts (Babylonian cuneiform tablets, Egyptian papyri, etc.) one can already find mathematical records, revealing a fine, subtle mathematics[2].

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[1] See S. Penrose, *The Road to Reality*.

[2] See H. Gericke, *Mathematik in Antike und Orient*.

There can be no doubt that such a development must have had a long history behind it, that is, that mathematics must have had its *roots*. Those roots lie hidden in the remote past with an almost total lack of material records until written history. To discover them we must thus refer to the biology of man, his faculties of thinking and speaking, his mythology etc., and to rely upon achievements in anthropology, archaeology, psychology, ethnology, linguistics, cognitive sciences, etc. An extraordinary progress in recent decades seems to justify such an attempt with a view to a sort of synthesis[1].

A terminological remark. Although the very term *mathematics* was introduced only by the Pythagoreans around the fifth century BC, embracing then *arithmetic*, *music*, *geometry*, and *astronomy* (also Greek terms), we do apply it also back, to preceding cultures, feeling justified by the fact that early *mathematical* texts reveal a resemblance to concepts and procedures observed in Greek and subsequent times. Up to the time of the Pythagoreans this particular kind of skill was unnamed, and treated as a kind of mystery, a useful know–how, a sort of knowledge available to a few initiated. By adopting a name, *mathematics* achieved, so to say, a self–conscience, soon to become an important component of Greek science. It should be noted, however, that the primary meaning of the word *mathematics* has evolved and that to this time we do not possess a commonly accepted definition of it. In consequence, we still proceed, as Pythagoreans did, by listing branches of what we call *mathematics*. *Arithmetic* and *geometry* are still there and so in this article we limit our attention to them, thus leaving apart *music* and *astronomy*.

Another remark. Mathematics is a knowledge which deals with some specific concepts and some specific relations between them. All the rest – mathematical statements and procedures (including deduction), further concepts and further relations – can be seen as an art which arose from those beginnings. However, analogously to the problem of understanding what mathematics is, there is a problem of understanding what a mathematical concept is. Leaving aside philosophical explanations (none of which is commonly accepted), one can only say that good mathematical concepts are both abstract in the extreme and somehow mysteriously related to the physical world. In this article we are more interested in the genesis of mathematical concepts than in the relevant philosophical questions but the latter cannot be omitted altogether.

Shaping mathematical concepts must have been a long process. Germs of those concepts surely have appeared long before their manifestations in the first historic civilizations and a long time should have been required for an elaboration of those germs to make them sufficiently abstract, general, and operational. Abstract to express features purged of physical substance, general to refer to objects or phenomena frequently met, and operational to allow an efficient treatment.

[1] The author is grateful to a referee for drawing his attention to some recent discoveries and to the editor for his efforts in replacing translations with the originals.

2. Evolution of Man

The evolution of Man is still unsatisfactorily explained, but it seems certain that prehuman beings separated from other primates ca 4 million years ago, that 2.5 million years ago they started to produce tools and to hunt, and that 1.5 million years ago there appeared *Homo erectus* with a brain twice as large as that of its predecessors and able to produce stone tools chipped on both sides[1]. Thus first there was an upright posture, then (stone) tools and hunting, and only after a big brain. Around 100 000 years ago appeared *Homo sapiens* and archaeological findings show that the last 40 000 years were a period of fast development and rapid changes.

The prehistory of man is usually divided into three great periods – stone, bronze and iron – taking their names from the basic raw material used to produce tools. The Stone Age was the longest one, lasting around 2.5 million years, and archaeologists divide it into shorter periods – Paleolithic, Mesolithic and Neolithic – and those into still further ones. For the present article only the Upper Paleolithic is meaningful, with its original cultures like Orygniac, Solutrean, and Magdalenian, created by *Homo sapiens fossilis*. It began (in our part of the world) around 40 000 years ago and lasted until 8000 BC[2]. We see there a half–settled mode of life, exquisite tools (spears, bows, etc.), rock paintings, and ornaments with geometric motives. In the Mesolithic (ca 9000–3500 BC), as a result of the withdrawal of the last glaciation, there were profound climatic changes, flora and fauna changed basically, husbandry took on a hunting–fishing–gathering character, there appeared nets and boats, and to produce tools flint was excavated in mines. In the Neolithic (ca 6000–2000 BC) changes were so rapid and profound that they are usually called the Neolithic Revolution: there agriculture and the raising of animals began, the first cities were built with their complex social structures, there appeared new handicrafts like weaving, and magic flourished. The discovery of bronze (ca 3500 in Mesopotamia and Egypt) marked the beginning of a new period with intensification of farming and animal husbandry. Trade was developing, while cities and societies grew larger. The last archaeological period is connected with the discovery of iron (ca 1500 BC), the accessibility and commonness of which has allowed a still more rapid further rise of farming (iron ploughs, axes, etc.) and of arms.

Since cultures with clear specific features could be observed only in the Upper Paleolithic, it seems reasonable to admit that it was only then that we meet a deeper mental life of primary communities. Specific types of behaviour (rites, skills) then observed were the ground upon which first the proto–mathematical concepts could have emerged. Consequently, it was only the Upper Paleolithic that was to see what might be called proto–mathematics, the Mesolithic to see the beginnings of arithmetic and geometry, and the Neolithic that was to leave the first written testimony of their development. In that long

[1] See S. L. Washburn, *The Evolution of Man* & S. L. Washburn & E. R. McCrown (eds.), *Human Evolution ...* .

[2] See J.–B. Duroselle, *L'Europe. Histoire de ses peuples*.

span of time man was still largely preoccupied with the struggle to survive and so his mental evolution still ran rather slowly. Only the Neolithic gave more spare time to reflect and to transmit discoveries to other people, and from that time on the mental evolution of man has gained momentum.

Upon that archaeological landscape we will now look closer upon the mechanisms of the mental evolution of Man.

3. Biological sources of mathematics

For a better understanding of how mathematics began, we distinguish three levels: biological, mental (psychic), and cultural (social).

Of the three, the most primary is the biological level. At that level the problem of sources has a somatic character, being related to the *central nervous system* consisting of brain and spinal cord. The deepest roots of mathematics seem to lie in the specific character of the reception by the system of some stimuli to what apparently the first to pay attention was R. Thom[1]. According to him, the system provides its owner with a specific *map* producing his/her physical surrounding. The map has the innate ability to:

a) recognize some forms and classifying them into *good*, *bad* or *neutral* ones;

b) precisely localize distinguished forms;

c) solving complicated dynamic problems, thus allowing prolonged operations like pursuit or flight.

In other words, the fundamental task of the central nervous system is to recognize basic spatial forms (some anthropologists call it the ability to form categories) and to stimulate basic interrelations, appearing in movements of living beings and inanimate nature. Of course, also animals are able to fulfil that task, but there is a basic difference. In the case of animals, responses to stimuli are spontaneous, almost hallucinatoric, while man has developed an ability to free himself from that hallucinatoric power. To understand it, we now come to the next level.

4. The faculty of thinking and speaking

The biological level naturally upgrades to the mental one. With the course of time anthropoids have developed the faculty of thinking, thus amplifying the ability of simulation and solving problems, and the faculty of speaking, as a tool of expression and contact. As vertical stature and tool production were characteristic of the early phases of human evolution, so the physiological faculties of thinking and speaking became the basis for the further development.

Originally the speech, as we see in animals and children, expressed only emotions (hunger, aggression, fear, satisfaction, etc.). Thinking did not possess a verbal counterpart (a monkey is able to take a stick to knock down the desired fruit, but it is utterly unable to express that sort of activity in understandable sounds). The latter view is corroborated by the recent

[1] See R. Thom, *De l'icône au symbole*.

discovery that numerical concepts have an ontogenetic origin and a neural basis which are independent of language[1].

Enormous progress in the study of relationships between language and numerical cognition, based mainly upon observations of indigenous groups that have a few or none number words, has been gained recently. These observations also lead to a conclusion that some calculation is possible without language (which means that arithmetic is prior to language).

The lines of thinking and speaking, initially separate, have intersected each other only in the process of hominisation, thus initiating a new, characteristic for man, way of behaviour: speaking acquires intellectual trait, while thinking becomes verbal[2]. In the affluence of observed facts and relations man has become able to distinguish and due to the faculty of speech also to name those of them which have been appearing more often, repeatedly and apparently were stable. By giving them names man has begun to free himself from their hallucinatoric power, while the just arising language reflected, in its syntactic structures, some typical interactions, thus allowing a simulation on a higher level. For instance, biological predation (cat–catches–mouse) can be seen as a paradigm for a transitive sentence (subject–predicate–complement). There followed a transition from emotionalism to rationalism and speech gained the level of language, an important instrument of analysis and communication[3]. Semantics reflected the world of form and phenomena, while syntax permitted expressions of some basic time–spatial interactions in movements of inanimate matter and of actions of living beings (river flows, hunter throws a spear). In that way speech paved the way to a rational reflection and thus to some understanding the world, a consequence of which were future discoveries of elements of higher order behind the phenomena appearing in the mind of man[4].

Looking back upon that early stage of forming the intellectual potential of man, one must be careful to avoid an ahistoric approach. It should be remembered that at these early beginnings the language was still largely primitive and inefficient, that the first mental images were naturally very simple and conceptions often false, and that the overall picture was rather obscure. Taking this into account, one should rather marvel at the fact that after all man has become able to proceed and thus to awaken himself (in a sense) to a deeper understanding of the world surrounding him[5]. In the next sections we shall describe some circumstances leading to that effect and how we have learned about it.

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[1] See R. Gelman & B. Butterworth, *Number and language: how are they related?*

[2] See L. Vygotsky, *Thought and language*.

[3] On that role of language see F. de Saussure, *Course in General Linguistics*.

[4] See L. J. Rips, *The psychology of Proof* ... & S. Carey, *The Origins of Concepts*.

[5] From the vast literature on the subject see the popular book: B. Butterworth, *The Mathematical Brain* & S. Dehaene & E. Brannon (eds.), *Space, Time and Number in the Brain*.

5. Symbolic thinking

Language – together with its semantics describing the physical reality full of specimens, with its syntactic dealing with the dynamic of that reality, and with other structures we only start to suspect – has become a constitutive element of man[1]. It was a necessary precondition for the transition from animality to humanity, from nature to culture. Language rendered possible a feeling of extraneousness to one's body with respect to the physical surrounding (an animal does not possess that feeling of strangeness) and, consequently, an ability to distinguish himself/herself as an element of a social group.

Language reveals the ability of thinking in symbols. The spoken word can be viewed as a sign replacing a physical object with a system of sounds. Sound signs have kept their meaning up to now, but their transitoriness led to the appearance of more durable, material symbols, which in turn evolved into writing. A man is able to distinguish a stimulus and its sign, a symbol and its designate. Animals react to stimuli but do not create and do not use symbols. In order to use symbols, it is necessary to distinguish symbol and object and to realize a certain relation between a sign and an object: *This is the cardinal characteristic which distinguishes human thought from animal response – the ability to distinguish A from B while at the same time recognizing that A and B are somehow interdependent.*[2]

Symbolic thinking made it possible to perceive analogies, to name them, and to designate them, with a big role played in this process by metaphors[3]. It also led to the autonomy of sign and to the replacement of biological simulation by a symbolic one. These were the main factors leading to the formation of a spoken common language and to giving it an internal dynamics with a strong internal tendency to generality. This in turn enabled a more subtle simulation, requiring an additional intellectual effort, but rewarding us with a still greater generality.

One of the great findings of the cognitive sciences is that our ideas are shaped, to a large extent, by our bodily, sensory–motor experience. And so the question arises: is the system of mathematical ideas also grounded (indirectly) in bodily experience? The answer seems to be positive and a bridge from our bodily experience to mathematical thinking seems to be provided by metaphors: mathematics, as we know it, arises from the nature of our brain and our embodied experience, and a great many mathematical ideas are inherently metaphoric in nature. In particular, such a fundamentally metaphorical idea is that of actual infinity. All forms of actual infinity are special cases of just one Basic Metaphor of Infinity[4].

Looking from the biologic–mental level, mathematics began with a representing processes simultaneously often and stable, thus tending to *a simulation of transition between processes undetermined and determined to*

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[1] An opinion held since antiquity. See Aristotle, *Parts of Animals* 660a.

[2] E. Leach, *Claude Lévi–Strauss*, p. 44.

[3] See C. Lévi–Strauss, *Le totémisme aujourd'hui*.

[4] See G. Lakoff & R. Nuñez, *Where Mathematics Come From*.

the utmost[1]. We will come back to this in section 11 below, allowing now a remark only that between common language and mathematics there is a difference of degree and not of nature. But it is exactly that degree which makes the difference.

An important element of the mental development of man was also memory, both individual and tribal. Individual memory enabled learning, while language and tribal memory have offered a chance to transfer accumulated knowledge and skills, that is, teaching. In particular, tribal memory allowed us to collect sub–territories important for the tribe, thus providing a sort of a narrative map of a larger region[2]. Thus began a process of the intellectual embracing of larger and larger regions of the world and of finding a greater and greater diversity of forms and phenomena, a process fundamentally connected to an internal dynamics of language, an important element of which was the semantic clearing of concepts away from layers of hard facts, an intellectual overtaking of the world: *Although man was surely not mute for most of his development, an increasing capacity for verbal communication may have been the ability that led to the extraordinary spread of modern man,* Homo sapiens sapiens.[3]

Language and symbols are connected with a social life, the basis of which is the possibility of communicating and transferring information. There is a sender sending a sign and there are recipients who read it. Social life was both a condition of and a stimulus to a further development.

6. Biogenetic principle of parallelism

In learning the intellectual history of man a great help is provided by the biogenetic principle, called also the principle of parallelism[4]. Shortly, it says that ontogeny is a recapitulation of phylogeny, which means that the development of each individual is a repetition of the evolution of the whole species. Considering the principle not only on the level of somatic development (as did Haeckel, its author) but also with respect to a mental development we may say that the mental development of each human being repeats that of mankind. In particular, the evolution of intellectual possibilities and behaviors of a child offers an insight into early stages which mankind has passed through.

Such a broader understanding of the biogenetic principle implies a methodological directive demanding to look after mutual relations between the

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[1] R. Thom, *Les mathématiques et l'intelligible*, pp. 74–75: *On peut dire, grosso modo, que la mathématique est née du besoin de simuler d'alternance entre processus indéterminés et processus extrêmement déterminés (convergents).*

[2] One can see such a map of a Mediterranean region in Homer's *Odyssey*.

[3] S. L. Washburn, *The Evolution of Man*, p. 206.

[4] The principle has been formulated by Ernest Haeckel (1834–1919) in 1886. It has gained a credit for being successful in zoology. Among advocates of its extended meaning and of its application to mathematics were H. Poincaré, *Science and Method*, G. Pólya, *The Teaching of Mathematics and the Biogenetic Law*, R. Thom (the author heard him expressing that opinion in his Paris seminar on philosophy of mathematics in the late 1980s), and some others.

development of a species (phylogeny), the individual development (ontogeny) and the systematic development as seen in the sequel of cultures. According to that directive, contemporary primitive tribes may also offer a valuable information about the evolution of material and intellectual human culture.

7. The intellectual development of a child and some phylogenetic consequences

The development of thinking and speaking runs, in the beginning of its life, along different lines. Soon, however, comes that decisive moment (usually in the second year of life) when the two lines meet. It is a turning point, the greatest discovery of a child so far: each object has its name. The child feels a strong need to express itself in words and tends actively to master a sign corresponding to an object, the sign of naming and communicating. There follows an unusually fast increase of the store of words. The child becomes conscious of the symbolic function of speaking, thus disclosing a comprehension of relations between a sign and its meaning.

The first to observe that the intellectual development of a child runs through qualitatively different stages was the Swiss psychologist Jean Piaget (1896–1980)[1]. He distinguished in that development three stages: the sub–operational stage, followed by the stage of concrete operations, followed by the stage of formal operations. Their names reveal their meaning. Since that time scholarly views have largely evolved[2], but Piaget's basic thesis – that of distinct qualitative steps – has never been questioned. Rather on the contrary, it has been successfully tested, sometimes reformulated, deeper documented.

Very interesting from that point of view were investigations of a Russian psychologist Lev Vygotsky (1896–1934). He also distinguished several stages in the intellectual development of a child but characterized them differently (syncretic thinking, analytic thinking, abstract thinking) and in each found several sub–stages. His investigations are all the more interesting because he offered a detailed description of the evolution of conceptual thinking, in particular the process of freeing words from the power of the concrete and through perceiving new connections achieving more general concepts, ending in a jump to abstract thinking.

In conclusion one may say that the phylogenesis of abstract concepts and of abstract thinking was a multistage, difficult and complicated process. As a child, who achieves the level of abstract thinking rather late, only as an adolescent teenager, mankind also came to it late and with apparent difficulty. And as a child who, despite mastering the highest form of thinking, does not resign from more elementary ones, also a modern man, to a large extent and in vast regions of his/her activity, does not always think abstractly.

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[1] See J. Piaget, *Genetic Epistemology* & J. Piaget, *The Language and Thought of the Child*.

[2] See K. Lovell, *The Growth of Basic Mathematical ...* , J. Piaget & B. Inhelder, *La psychologie de l'enfant*, M. Donaldson, *Children's minds* & E. Spelke, *Differences in Intrinsic Aptitude for Mathematics and Science?*

8. Mythology, magic, rites

Biological and mental levels offer a proper perspective to the beginnings of mathematics, but on mathematics itself they do not speak much. With that end in mind, we should now consider the third level, a cultural one. Mathematics is a part of culture and so to study its beginnings – how basic mathematical concepts have emerged and were shaped and how basic relations between them were found – there is no other way but to start with the culture, both material and mental. Data, however, are rather scarce, because the majority of scholars interested in old cultures restrict their attention to their material side, which may lead to a false impression that the mental activity of prehistoric people was essentially limited to the preservation and development of technology. This omits an enormous sphere against which Mircea Eliade (1907–1986) protested vehemently, writing that [s]*uch an opinion is not only erroneous, it is fatal to a knowledge of man.* Homo faber *was at the same time* Homo ludens*,* sapiens*, and* religiosus.[1]

Primitive man, freeing himself through the medium of language from the motorial slavery of physical neighbourhood, fell into the slavery of primitive images and beliefs. This can be seen in his/her ritual approaches to the surrounding world, particularly with respect to animals and plants, the diversity and interdependences of which were for him/her the first model of social structures. Stimuli, which previously had mostly physical and individual meaning (as a rule leading to a motorial response), now begun to take on a group meaning, strongly influenced by magic. Mythology, magic and rites became characteristic elements of human civilization in that time. Due to a system of beliefs putting some order into the accumulated knowledge and to rites allowing for the *taming* of nature and gaining some profits from its phenomena – one could survive. Myth has become

> *a narrative resurrection of a primeval reality, told in satisfaction of deep religious wants, moral cravings, social submissions, assertions, even practical requirements. Myth fulfills in primitive culture an indispensable function: it expresses, enhances, and codifies belief; it safeguards and enforces morality; it vouches for the efficiency of ritual and contains practical rules for the guidance of man. Myth is thus a vital ingredient of human civilization; it is not an idle tale, but a hard–worked active force* [...].[2]

There begun a culture, the oldest material expressions of which were burials and ornaments. Mythology, magic and rites offered everybody a feeling of participation, confirming or changing his/her status in the group (e.g. rites of initiation, marriage, burial), explaining the surrounding world (see myths of beginning, different deities) and providing instruments of influence (casting a spell, offering a sacrifice, other magical interventions). A

[1] M. Eliade, *A History of Religious Ideas*, vol. 1, p. 8.

[2] B. Malinowski, *Myth in primitive psychology*, p. 79.

man of such a culture was deeply immersed in the word of its signs and values. Some myths were known already in Paleolithic, mostly cosmological ones and myths of beginnings: *The immense dissemination of this cosmogony and its archaic structure point to a tradition inherited from earliest prehistory.*[1]

Mythology, magic and rites reveal the dominant experience of *sacrum*, that is, the experience of a reality existing beyond a man and independent of him/her but at the same time providing a meaning to his/her existence and offering a basis for its explanation. As M. Eliade has put it,

> *"it is difficult to imagine how the human mind could function without the conviction that there is something irreducibly* real *in the world; and it is impossible to imagine how consciousness could appear without conferring a* meaning *on man's impulses and experiences. Consciousness of a real and meaningful world is intimately connected with the discovery of the sacred. Through experience of the sacred, the human mind has perceived the difference between what reveals itself as being real, powerful, rich, and meaningful and what lacks these qualities, that is, the chaotic and dangerous flux of things, their fortuitous and senseless appearances and disappearances".*[2]

In other words, every *manifestation of the sacred is important: every rite, every myth, every belief or divine figure reflects the experience of the sacred and hence implies the notions of* being*, of* meaning*, and of* truth.[3] It was the soil to gave birth to some abstract conceptions related to the world, including proto–mathematical ones.

The way in which it could be done has been suggested by J. G. Fraser's analysis of thinking systems upon which magic and rites have been resting. According to him, there are two basic principles[4]. The principle of similarity says that similar causes similar, that is, one may gain an effect by imitating it on a similar object (e.g. to provoke the death of an enemy by destroying his image) or by imitating the effect itself (e.g. restoring to health). The principle of contagion says that *things which have once been in contact with each other continue to act on each other at a distance after the physical contact has been severed*[5] (e.g. the lock of hair of a beloved person). Although *magic is a spurious system of natural law as well as a fallacious guide of conduct*[6], it was undoubtedly a necessary stage, as is confirmed by its

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[1] M. Eliade, *A History of Religious Ideas*, vol. 1, p. 26.

[2] M. Eliade, *A History of Religious Ideas*, vol. 1, p. xiii.

[3] M. Eliade, *A History of Religious Ideas*, vol. 1, p. xiii.

[4] See G. Fraser, *The Golden Bough*.

[5] G. Fraser, *The Golden Bough*, p. 52.

[6] G. Fraser, *The Golden Bough*, p. 53.

existence in all known primitive tribes. One should however perceive and appreciate in it an effort to understand and to survive in the alien and ominous world. Let us also notice that to understand the world – on that level of primitive magic – one should accept already then that *things act on each other at a distance through a secret sympathy, the impulse being transmitted from one to the other by means of what we may conceive as a kind of invisible ether*[1]. Magic testifies also to an acknowledgement of the fundamental homogeneity of our world (similar causes similar), thus opening the way to further abstractions.

9. Counting and the beginning of arithmetic

There has been a prevailing opinion for a long time that the beginnings of civilization are connected with rites[2]. In those beginnings, full of myths, rites and misconceptions, there appeared ideas which led, in due time, to notions of number, dimensions, and figure, a starting point for mathematics (arithmetic, geometry). This is the beginning of proto–mathematical conceptions.

Numbers are necessary to reasoned human existence. Proto–mathematical germs of the number conception can be seen in the sacral rites and ludic activity of primitive man. According to the Pavlov's principle of unconditional reflexes, any successful human action yields a tendency for it to be repeated, even in the absence of the initial conditions. In rites, which were an important part of primitive culture, some simple elements were met repeatedly. Emancipated from the primitive context, they could serve as the beginning for something new. According to A. Seidenberg[3], the beginnings of counting (and, consequently, of numbers) are hidden behind the ritual of creation. To count we need a series of words and an activity, sufficiently common, to which the series could be applied. The ritual of creation has offered both. Following A. Seidenberg it might thus happen that names of the consecutive participants in the ritual (or the words announcing them) were the first number names. As time went on, such a *counting* could break off from the ritual and take on a more general character of counting.

An interesting discovery is that early counting hardly resembles later arithmetic. For instance, [a]*lthough the Mundurukú lack words for numbers beyond 5, they are able to compare and add large approximate numbers that are far beyond their naming range. However, they fail in exact arithmetic with numbers larger than 4 or 5*[4]. Thus there is a distinction between a nonverbal system of number approximations and a language–based counting system for exact numbers and arithmetic.

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[1] G. Fraser, *The Golden Bough*, p. 54.

[2] See F. R. S. Raglan, *How Came Civilization* & S. Dehaene, F. Spelke, P. Pinel & R. Stanescu, *Sources of mathematical thinking ...* .

[3] See A. Seidenberg, *The ritual origin of counting* & A. Seidenberg, *The origin of mathematics*.

[4] P. Pica, C. Lemer, V. Izard & S. Dehaene, *Exact and approximate arithmetic in an Amazonian indigene group*, p. 499. See also V. Izard, *Interactions entre les représentations numériques verbales et non–verbales ...* & S. Dehaene, V. Izard, C. Lemer, P. Pica, *Quels sont les liens entre arithmetic et langage?*

To similar conclusions came Daniel Everett who explored the language of Indians Pirahã in Amazonia, never studied before. He found that Pirahãs have no counting system neither qualifiers equivalent to *all* or *each* but are able to deal with approximate quantities[1].

Modern linguistic analysis may offer another insight into those remote beginnings. Exploration of a generative grammar may lead to the discovery of universal features and of the formal organization of numeral systems[2]. Still another insight is provided by modern psychology[3].

Language expressions lead to the conclusion that the idea which led to the conception of number had a qualitative character initially. Early number names were a part of common language, and language traces show that that qualitative character has lasted for a long time. It was *that man* and neither *the first man* nor *one man*. Also a *pair* was distinct from *two*. Such traces can be still observed in languages which possess articles and dual number besides the plural one, e.g. in Greek, Celtic and some others.

Another question may arise: were these primeval numbers cardinal (meaning: amount) or ordinal (meaning: succession)? Surely both meanings were present, for the numbers 1, 2, 3, ... assume an order from lesser to bigger and the number n+1 means both the next to n and some amount.

Some numbers have kept for a long time an individual mystic meaning (like *7* or *13* even nowadays), which was a serious obstacle to their general understanding. Also contemporary primitive tribes disclose a great variety of ways of counting and naming numbers, and of range of operations to deal with them[4].

Concluding, we may say that:

a) richness of number names can be related to the fact that for a long time counted objects were not clearly distinguished from their names; a primitive man was able to grasp mentally a bigger amount of objects (e.g. a group of hunters) without counting them;

b) the process of generalization of the number concept was a long one and surely passed through several stages;

c) also the mystic meaning of some numbers has lasted for a long time, e.g. ancient Greeks had given to numbers 1, 2, ... , 10 some mystic meaning[5]; such an additional meaning rendered adding or dividing difficult, thus making primeval numbers hardly operational;

d) there is no *natural* basis for counting (in use were 2, 4, 5, 10, 20, 60; e.g. in Babylonia it was 60, in Rome 5 and 10, while French language keeps remnants of 20);

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[1] See D. Everett, *Don't Sleep, There are Snakes ...* .

[2] See J. R. Hurford, *The Linguistic Theory of Numerals* & Th. Crump, *The Anthropology of Numbers*.

[3] See H. Wiese, *Zahl und Numerals*.

[4] The Tsimshian Indians from British Columbia have seven sets of words to denote the amount of animals and plane objects, of time and round objects, of men, of long objects, of boats, of measures, and of other things. See M. Moffat, *The Origins*. For some recent discoveries see: G. Saxe, *Cultural Development of Mathematical Ideas* & G. Urton, *Signs of the Inka Khipu ...* .

[5] See B. L. van der Waerden, *Erwachende Wissenschaft*.

e) in spite of all of that a primitive man was able to work out some rules to deal with numbers (although those rules were undoubtedly more complicated and weaker than ours based upon abstract concepts); it was a great step forwards in comparison to an earlier inoperative stage.

The road, made by prehistoric man, from a ritual dance to arithmetic and geometry of the first historic civilizations, was long and embarrassingly troublesome. It must be held in respect.

10. Measuring

Somewhere on that road from rites to mathematics there appeared measuring. All tribes we know do measure. They measure time, length, width, depth, weight, etc. Considering counting as a measure of amount, we have a genetic connection: counting was first, measuring next. And similarly to labeling numbers to count, numbers to measure also were labeled, mainly by parts of human body: fingers, feet, ells, etc.

The genetic connection between counting and measuring was a bridge joining numbers to figures, testifying to the generality of the number concept and stimulating primitive arithmetic: lengths, weights, etc. can be naturally added, while mystic numbers hardly can.

11. The sequence of natural numbers

The conception of an abstract sequence 1, 2, 3, ... emerged relatively late. It took a long time to apprehend that counting was essentially a procedure leading from a given set to its enlargement by one element (one fish more, one day longer), that is, from a natural number n to the natural number $n+1$. The commonness of that procedure, and the ludic tendency to repeat and the internal dynamic of language have eventually led to the recognition of the infinite sequence 1, 2, 3, ... of all natural numbers, detached from any physical context.

Revoking Thom's idea (section 5) that mathematics had begun with the *simulation of a transition between processes undetermined and determined to the utmost*, one can say that the here undetermined part of the process is number n, while the deterministic is the operation +1:

> *A characteristic feature (deeply inconsistent with nature, as Dieudonné justly remarked) is that the operation "adding one" is always qualitatively the same, not in the least depending on previous operations which led to the determination of the value* n *upon which it presently acts. It happens as if the result of each operation was "kept in the memory", being there totally neutralized until operation* +1 *is applied. Only while determining the result of that operation does the sense* n *from the memory appear in the value* n+1 *received after the operation. No*

natural system does or will ever provide a model isomorphic to the set of all natural numbers.[1]

Other examples of the simulation of a transition from indeterminacy to determinism: arithmetical addition of two natural numbers, a free monoid with two generators[2].

12. Beginnings of geometry

Kant has argued that Euclidean geometry is synthesized on the basis of an a priori intuition. Recent research based upon studies of an indigenous group in Amazonia seems to support that view. There is an evidence for primary geometrical intuitions in the absence of education among Munduruku[3], and even more, that *during childhood humans develop geometrical intuitions that spontaneously accord with the principles of Euclidean geometry*[4].

Similarly to numbers and arithmetic, rites led also to distinguishing and naming some simple geometric figures such as segment, circle, square[5]. Originally they were sacral figures appearing in ritual dances but after their naming the process of extending their meaning was inevitable, e.g. a circle in dance could also present the sacral trajectory of the sun in the heavens. The commonness of solar cults shows that the sun was one of the most popular deities and that some knowledge about its trajectory possessed great practical value, allowing the tribe to adapt to its regularities (day: hours, year: seasons). In turn this led to a primeval calendar, that is, to counting cyclical time (essential here are solstices, both summer and winter) and to primeval astronomy, that is, distinguishing shapes in constellations and describing the trajectories of celestial bodies. These physically inaccessible regions have thus became open to a penetration by reason, opening new intellectual horizons and inviting people to more abstract problems.

The sacral character of the first proto–mathematical conceptions is also present in the Indian conviction that an altar must be a precise realization of a sacred figure, and that otherwise prayers could be ineffective.

Related to geometry is also the concept of space, emerging from the way in which a primeval man understood his closest surroundings. For him/her that surrounding was full of meaning. According to L. Lévy–Bruhl each inch of ground *is determined by its appearance, form, rocks present there, trees, waters, sands, etc. Even more, it is mystically related to visible and invisible beings, which have appeared there or still are there* [...] *and it would not be what it is without them, neither they without it.*[6] Interrelations between that

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[1] R. Thom, *Les mathématiques et l'intelligible*, p. 75.

[2] See R. Thom, *Les mathématiques et l'intelligible*, p. 76.

[3] See S. Dehaene, V. Izard, P. Pica & E. Spelke, *Core knowledge of geometry in an Amazonian indigene group*.

[4] See V. Izard, P. Pica, S. Dehaene, D. Hinchey & E. Spelke, *Geometry as a Universal Mental Construction* & V. Izard, P. Pica, E. Spelke & S. Dehaene, *Flexible intuitions of Euclidean geometry ...* .

[5] See A. Seidenberg, *The ritual origin of geometry*.

[6] L. Lévy–Bruhl, *Fonctions mentales dans les sociétés inférieures*, p. 130: *En outre, dans la portion de territoire ainsi définie, chaque localité caractérisée par son aspect, par sa forme, par tels rochers, tels arbres,*

surrounding and a man are so strong that L. Lévy–Bruhl has elevated them to the fundamental principle of participation, that is, joint participation of beings and of objects within a collective image.

Although territorially limited, the nearest surroundings of a man are vivid and full of variable meanings, which influence their shape and properties. Using modern language, one could say that topology of those surroundings is rich and variable. And were it not for a contact with the *sacrum* of the celestial sphere, suggesting a deeper reality, it could stay as such.

The contrast between that primeval image of a surrounding space and the geometric space of ancient Greeks, as expressed in Euclid's *Elements*, testifies to an enormous work which has been done to clean up the primeval topology of unnecessary meanings to simplify (that is, to impoverish original semantically rich concepts) and to make it independent of time. All that work was done in prehistoric times. Even very primitive tribes possess some knowledge of the movements of sun, moon and stars, and it is precisely that knowledge which underlies the beginnings of abstract conceptions of a sphere, straight line, angle, and, more generally, of geometry.

13. Problem of priority: arithmetic or geometry?

One of the more interesting problems in the prehistory of mathematics is: which was earlier, geometric figures or arithmetic? On the side of the ontological priority of geometry, based on the conviction of the dominating role of a spatial image created by the central nervous system, is R. Thom: *Euclidean space pre–exists in formation of our mental activities.*[1] On the other hand, a reconstruction of the ancient Indo–European language, based upon the oldest historic languages in that group, shows the presence of numerals and absence of geometric figures (with the exception of the wheel).

14. Material testimonies

Due to the efforts of archaeologists material testimonies related to number and coming from the Paleolithic are a great many, reaching tens of thousands[2], and there are emerging still new ones. They testify to the existence of symbols and magic rites, revealing a new tendency to decoration and ornaments which was a further stimulus to develop mathematical thinking.

One of the oldest material relics of the past, which we could point to, is an ox's rib over 200 000 years old with interesting cuts, made by a man with the brain twice as small as that of Neanderthal man. He made two parallel incisions, then again two parallel incisions at an angle to the previous ones, and repeated the procedure several times, each time with a different tool. We

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tel point d'eau, telle dune de sable, etc., est mystiquement unie aux êtres visibles ou invisibles qui s'y sont révélés ou qui y ont leur séjour [...] *elle ne serait pas sans eux ce qu'elle est, ni eux sans elle.*

[1] R. Thom, *Les mathématiques et l'intelligible*, p. 77.

[2] See B. A. Frolov, *Czisła v grafikie paleolita*. The book is richly illustrated and contains a vast bibliography of nearly 500 items.

do not know why he did it, but undoubtedly it was a conscious work, done over a long period and close to counting[1].

A still wider picture of a spirit life could be observed in tribes living in the Upper Paleolithic, including the Cro–Magnon people who appeared around 35 000 years ago and ousted the Neanderthal people. Europe in that time was still largely covered with a glacier and the sea level was lower by about 100 m. The tundra, forest and steppes, freed from the glacier, were rich in animals. Cro–Magnon hunters were settling along rivers (mainly in the south of present France), seeking refuge under rocks and in caves. After some 25 000 years the warming of the climate caused the melting of glaciers and the sea level was raising. Invasion of the forests had ousted big steppe animals (mammoths then died off). The Neolithic came with some farming and settled life.

Extremely interesting is the so–called Blanchard bone, ca 30 000 years old, which contains 69 signs representing results of an observation of moon phases during 2 and ¼ month[2]. The bone shows that a Cro–Magnon man was already aware of passing the time, was capable of making long lasting observations and recording their results with symbols designed for the purpose.

Another bone 15 000 years old contains long series of cuts[3]. Perhaps it was a sort of calendar. In any case the man who did it and its later users had the ability to count and to note down results (numbers).

Even more, *since hunter of the glacial epoch has reproduced plant species from different year seasons as precisely as animals and their behavior, and since he had used to that end drawings both abstract and realistic, he most probably had the words to give a name to these differences and processes*[4]. In other words, Cro–Magnon men possessed an ability to speak and one may risk an opinion that already in that early period of culture of *homo sapiens sapiens* we see the beginnings of distinguishing and naming, abstracting and symbolizing.

Megaliths in this respect are fascinating monuments[5]. Precise measurements made by the carbon 14 dating show that they were built 4000–2000 BC, which means that some are older than the Egyptian Pyramids or Babylonian ziggurats. In particular, Stonehenge was built 2900–2200 BC (most interesting phase I), 2100–2000 BC (phase II) and 2000–1100 BC (monumental phase III). Its main axis points at the direction of the summer solstice which shows that the construction was a sort of calendar, perhaps an observatory of the moon, its phases and movement, and perhaps even a calculator to find lunar and solar eclipses[6]. The latter hypothesis, however, seems to be rather

[1] See A. Marshack, *Exploring the Mind of Ice Age Man* & A. Marshack, *The Roots of Civilization ...* .

[2] See A. Marshack, *Exploring the Mind of Ice Age Man* & K. Overmann, *The Role of Materiality in Numerical Cognition*.

[3] See A. Marshack, *Exploring the Mind of Ice Age Man*.

[4] A. Marshack, *Exploring the Mind of Ice Age Man*.

[5] See A. Thom, *Megalithic Sites in Britain*, D. C. Heggie, *Megalithic Science* ... & M. Parker Pearson, *Stonehenge*.

[6] See G. S. Hawkings, *Beyond Stonehenge* & F. Hoyle, *On Stonehenge*.

weakly supported and more sceptical scholars are rather inclined to the view that Stonehenge was a sacral building with some elements of astronomy[1].

A closer examination of stone circles in Scotland, with the characteristic rock on the periphery, embraced on both sides by vertical rocks, leads to the hypothesis that they served as spectacles with the main role played by moving full moon. Construction was too rough to be useful for precise astronomic observations, but

> *it is ideal, however, for people engaged in long ceremonies performed by moon light. At Sunhoney, for instance, it would have taken the moon more than an hour to cross the space between the two flanking stones, its light shining all the while past the recumbent onto the ring coins* [...] *So dramatically must the passing moon have seemed framed between the silhouetted flames that this may well have been the chief effect sought by the builders.*[2]

Circles were not cemeteries, although some of them reveal remnants of people killed there. All of it – moon, fire, death – shows its ritual character.

Some recent results point also to some other ideas in the culture of present primitive tribes, which most surely were present in prehistoric times[3]. And linguistic anthropologists have much to say about the origin and evolution of numeracy, the cognitive and cultural foundations of numbers, e.g. K. Overmann offers a reinterpretation of the cultural nexus of Middle Stone, Upper Paleolithic, and Neolithic Ages[4], and S. Chrisomalis provides an analysis of items of written numerals as used in the past 5000 years[5]. For a broader view, see the book *Changing View on Ancient Near Eastern Mathematics*[6].

15. Number protonotation

Between prehistoric and historic people one may distinguish, in a region of present Iraq and Iran, a long transitory period, in which there appeared and were disseminated tokens to denote numbers and measures and perhaps also a sort of denoted objects. According to D. Schmandt–Besserat[7], they appeared in the ninth millennium BC and served until the end of the fourth millennium BC, when hitherto loose tokens have begun to be collected in clay containers protecting them, with marks denoting tokens and some additional information on them. In due time they evolved to cylinder seals and clay tablets with the number signs. And at the end of the fourth millennium BC there existed two

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[1] See R. J. C. Atkinson, *Stonehenge*.

[2] A. Burl, *The recumbent stone circles …* . See also A. Burl, *Stone Circles of the British Isles*.

[3] See M. Asher, *Ethnomathematics*.

[4] See K. Overmann, *The Role of Materiality in Numerical Cognition*.

[5] See S. Chrisomalis, *Numerical Notation …* .

[6] See P. Damerow & J. Hoyrup (eds.), *Changing View on Ancient Near Eastern Mathematics*.

[7] See D. Schmandt–Besserat, *The Earliest Precursor of Writing* & D. Schmandt–Besserat, *Before Writing*, vol. 1.

similar (but not identical) well developed number and measure systems, called proto–Elamic and proto–Sumerian[1]. They offer an interesting insight into the economy of those people (but we do not know whether they really were Elamites or Sumerians) and into their arithmetic. We find there the counting of people, calculating grain supplies to feed people or cattle, summing up temple gifts of bread and beer, marking deliveries to and from a granary, etc. Calculations with numbers amounting to thousands on the one hand and to fractions on the other have become easier by the adoption of a suitable system of measures and by the germs of the later 10–60 system.

From the viewpoint of the history of mathematics one may thus note that

a) primeval numbers are, as a rule, labeled, that is, they are not abstract but refer to a quantitative measure, e.g. two measures of oats or three measures of beer;

b) the same sign may have different meanings depending on context;

c) number signs are accompanied by other signs which means that alphabetization has come after numerization.

16. Problem of a common source

Historians have raised also the problem of a source: whether mathematics began in one place or independently in many? To the former point, we can adduce some astonishing similarities between mathematics in different regions:

a) A. Seidenberg discovered that Hindu altar constructions, as described in the *Śulvasutras*, used the *theorem of Pythagoras* and concluded that algebra and geometry in Mesopotomia, the *algebraic* geometry of Greece, and the geometry of India have a common source;

b) the Babylonian tablets Plimpton 922 or YBC 7289 show awareness of the Pythagorean theorem[2];

c) in megalithic constructions the *theorem of Pythagoras* also was used (in the form of triangle with the sides 3, 4, 5)[3];

d) B. L. van der Waerden has compared early Chinese mathematics (as documented in *Nine Chapters of the Arithmetical Art*) and early Babylonian mathematics, also he discovered many similarities.

Taking into account arguments such as those above, B. L. van der Waerden has tried to reconstruct neolithic mathematics of the period 3000–2500 BC (including the *theorem of Pythagoras*) and claimed that it expanded from its source somewhere in Mesopotamia to the British Isles, Near East, India, and China[4]. This claim, however, has been strongly criticized. Critics have pointed to the lack of really convincing arguments, advancing a more probable, in their opinion, hypothesis that mathematics reached some level of maturity in the third millennium BC in Egypt and Sumer independently and

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[1] See J. Freiberg, *Numbers and Measures ....* .

[2] See J. Freiberg, *Methods and traditions of Babylonian mathematics*.

[3] See E. MacKie, *Science and Society in Prehistoric Britain* & E. MacKie, *The Megalith Builders*.

[4] See B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations*.

from there it went to India, China and Greece. A visibly independent mathematics evolved in Pre–Columbian America.

At present a still more modest opinion seems to be in the air. Expressed by I. R. Shafarevitch more than half–a–century ago it says that mathematics results from individual efforts by many people dispersed upon all continents throughout all times[1]. Or, even more emphatically, *mathematicians don't make mathematics, they're instruments for mathematics to make itself*[2]. This opinion is close to the Platonic view that mathematics is a way to the discovery of the world of ideal forms.

17. Concluding remarks

Our knowledge on the beginnings of mathematics, as outlined above, allows us to express the following cautious conclusions:

a) the biological roots of mathematics lie in the specific structure and specific aptitudes of a central nervous system;

b) the psychic roots of mathematics lie at the cross–roads of speaking and thinking, mathematics being a peculiar kind of language, poor semantically (abstract to the utmost), but precise and offering unshakable conclusions;

c) primeval mathematical ideas seem to exist already 35 000 years BC; although unnamed until Greeks, mathematics was a part of human culture since remote times;

d) the evolution of mathematics was not linear: it went by degrees separated by long periods of negligence and oblivion, and not being bounded to any specific area, it could be taken up in another place, from another starting point, and with a different pace;

e) knowledge of numbers and calculations (*arithmetic*) and knowledge of figures and measuring (*geometry*) went parallel, problems of priority or (periodical) domination remain open;

f) number symbols appeared much earlier than word symbols, alphabetization begun in the process of describing the meanings of number tokens;

g) at the end of prehistoric era and in the dawn of written history mathematics is sufficiently developed to contain some highly abstract notions of numbers, figures and their interrelations;

h) the intellectual work done in prehistoric times was enormous, resulting in a solid base for further development; the work, however, is little known and remains underestimated.

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[1] See I. R. Shafarevitsh, *Über einige Tendenzen in der Entwicklung der Mathematik*.

[2] R. Hersh, *What Mathematics Is, Really?*, p. 85.

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